

# Asset Pricing and Welfare Analysis with Bounded Rational Investors

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Motivated by the fact that investors have limited ability to process information, I model investors' bounded rational behavior in processing information and investigate its implications for asset pricing. Investors can make mistakes in processing information and thus have inaccurate estimates of fundamentals. This process generates “bounded rational risk.” I find that the volatility of stock and bond return increases in the presence of bounded rational investors, which can help explain the excessive volatility puzzle. The stock-bond return correlation becomes time varying and even negative and rational investors benefit from the trading with bounded rational investors.

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## 1. Introduction

The magnitude of fluctuations of stock prices remains a challenge to explain. Shiller (1981) finds that stock prices move too much to be explained by changes in dividends or cash flows, and refers to it as the "excessive volatility." In bond market, Shiller (1979) finds that the observed bond yield volatility exceeds the upper limit implied by the expectation hypothesis and refers to it as the "excessive bond yield volatility" puzzle (e.g., also see Gurkaynak, Sack and Swanson (2005); Piazzesi and Schneider (2006)). Baele, Bekaert and Inghelbrecht (2007) find that the stock-bond return correlation was as high as 75% and dropped to lower than -60% between 1968 and 2004. This substantial time variation of the stock-bond return correlation is hard to explain by existing models.

Motivated by these facts, I build a general equilibrium model to explain the anomalous changes in stock and bond markets. In financial markets, investors face a wealth of information that requires a large amount of knowledge and ability to process. For example, investors can find voluminous information on the Internet, but it is hard for them to well exploit. Some investors can make mistakes in processing information and thus have inaccurate estimates of fundamentals.

I assume that investors have CRRA preferences, and choose consumption and portfolios to maximize their utility. To solve for the equilibrium, I construct a representative agent with stochastic relative wealth weights (e.g., Cuoco and He, 1994; Basak and Cuoco, 1998). I use the Clark-Ocone formula and the martingale representation theorem (e.g., Ocone and Karatzas, 1991; Nualart, 1995) to decompose the volatilities of assets and employ Monte Carlo simulation to compare the model with the benchmark.

The main results are as follows. First, the stock price increases in the presence of bounded rational investors when the drift of signal process exceeds dividend growth. Intuitively, when the drift of signal process is higher than dividend growth, bounded rational investors are more optimistic than rational investors since their estimate of dividend growth equals the drift of signal process. It in turn makes bounded rational investors perceive the stock more favorably and causes the stock price to increase. Second, volatilities of stock and bond returns increase. Both “learning risk” and “bounded rational risk” affect volatility of asset returns, making asset prices more volatile. Third, the stock-bond return correlation decreases significantly and even becomes negative. Finally, rational investors benefit from the trading with bounded rational investors.

This paper is related to at least two streams of literature. First, this paper can be compared to other papers that study investors’ irrational behavior. Dumas, Kurshev and Uppal (2009) provide a model in which there are two types of investors: one has correct beliefs about the signal and the other is overconfident in it. They analyze the implications of overconfident investors for asset pricing and hedging strategies. I investigate the time variation of stock-bond return correlation and rational investors’ welfare change in the presence of bounded rational investors. Berrada (2008) considers a model, in which some investors overreact and underreact to information. Overreactive (underreactive) investors allocate a high (low) weight to their updated beliefs, and thus stock volatility is increased (decreased). In my paper, due to investors’ bounded rational behavior, the distribution of wealth across investors changes over time and thus stock volatility is affected.

Second, this paper is connected to heterogeneous beliefs. Detemple and Murthy (1994), Zapatero (1998), and Basak (2000) investigate the impact of heterogeneous beliefs among investors on asset pricing. They assume that investors have logarithmic preferences, and thus the stock price and its volatility are not affected by heterogeneity in beliefs. Buraschi and Jiltsov (2006) and Xiong and Yan (2009) present models in which investors are rational and they investigate the effects of disagreement on option-pricing and bond pricing, respectively. This paper is also related to David (2008) in some aspects, although both investors are assumed to be rational in David (2008).

## 2. The economy

I consider a continuous time, pure exchange economy, in which two types of investors are endowed with an exogenous aggregate dividend. Uncertainty is represented by a complete probability space  $(\Omega, \mathcal{F}, P)$  and a two dimensional Brownian motion process  $W = (W_1, W_2)^T$  defined on  $(\Omega, \mathcal{F})$ . Letting  $\{F_t^W\}$  denote the augmented filtration generated by  $W$  and  $H$ , a  $\sigma$ -field independent of  $F_T^W$ . The complete information filtration  $\{F_t\}$  is the augmentation of the filtration  $H \times \{F_t^W\}$  (e.g., Basak, 2000).

### 2.1 Dividend process and information structure

I assume that the economy includes two types of investors  $i = A, B$ . They observe the same exogenous aggregate dividend process  $D$  which has the dynamics

$$d \ln D(t) = \mu_D dt + \sigma_D^T dW(t), \quad (1)$$

where dividend growth  $\mu_D$  is constant and unobservable.  $W = (W_1, W_2)^T$  is a two dimensional Brownian motion, and  $W_1$  and  $W_2$  are independent.

I assume that investors also observe a public signal,  $e$ , given by

$$de(t) = \mu_e dt + \sigma_e^T dW(t), \quad (2)$$

where the drift coefficient  $\mu_e$  is constant and observable, and  $\mu_e \neq \mu_D$ . The signal provides information about Brownian motion  $W$ , which affects the dividend process but does not provide information about unobserved dividend growth (e.g., Zapatero, 1998).

## 2.2 Rational investors' evolution of beliefs

Investors  $A$  are assumed to be rational. They know that the signal provides information about  $W$  but not about  $\mu_D$ , and update their estimate of  $\mu_D$  and its variance via  $\mu_D^A(t) = E^A[\mu_D | F(t)]$  and  $v^A(t) = E^A[(\mu_D^A(t) - \mu_D)^2 | F(t)]$  (e.g., Liptser and Shirayayev, 2001). They perceive the processes of dividends and signal as follows:

$$\begin{aligned} d \ln D(t) &= \mu_D^A(t) dt + \sigma_D^T dW^A(t), \\ de(t) &= \mu_e(t) dt + \sigma_e^T dW^A(t), \end{aligned} \quad (3)$$

where  $dW_1^A(t) = dW_1(t) + \kappa_1(\mu_D - \mu_D^A(t))dt$  and  $dW_2^A(t) = dW_2(t) + \kappa_2(\mu_D - \mu_D^A(t))dt$ , and

$$\kappa_1 \equiv -\frac{\sigma_{e1}(\mu_D - \mu_D^A(t))}{\sigma_{D1}\sigma_{e2} - \sigma_{D2}\sigma_{e1}} \quad \text{and} \quad \kappa_2 \equiv \frac{\sigma_{e2}(\mu_D - \mu_D^A(t))}{\sigma_{D1}\sigma_{e2} - \sigma_{D2}\sigma_{e1}}.$$

Under regularity conditions,  $\mu_D^A$  follows the dynamics given by

$$d\mu_D^A(t) = \kappa_2 v(t) dW_1^A(t), \quad (4)$$

where  $v(t) = \frac{1}{1/v(0) + \kappa_2 t}$ . It means that investors  $A$ 's estimates of dividend growth converges to actual growth as time  $t$  goes to infinity. According to the literature of incomplete information, “learning risk” is generated when investors update their beliefs.<sup>1</sup>

### 2.3 Bounded rational investors' evolution of beliefs

Investors  $B$  mistakenly believe that the signal provides information about dividend growth and assume that  $\mu_D = \mu_e$ . I call investors  $B$  bounded rational investors. From their perspective, the processes of dividends and signal follow the dynamics, given by

$$\begin{aligned} d \ln D(t) &= \mu_e(t)dt + \sigma_D^T dW^B(t), \\ de(t) &= \mu_e(t)dt + \sigma_e^T dW^B(t), \end{aligned} \quad (5)$$

where  $dW_1^B(t) = dW_1(t) + \kappa_2(\mu_D - \mu_e)dt$  and  $dW_2^B(t) = dW_2(t) + \kappa_1(\mu_D - \mu_e)dt$ .

Therefore, I have the following relation

$$dW^B(t) = dW^A(t) + \Phi(t)dt, \quad (6)$$

where  $\Phi \equiv (\Phi_1, \Phi_2)^T$  satisfies  $\Phi_1(t) \equiv \kappa_2(\mu_D^A(t) - \mu_e)$  and  $\Phi_2(t) \equiv -\frac{\sigma_{e1}}{\sigma_{e2}}\Phi_1(t)$ .

Since bounded rational investors make mistakes in processing information and thus have incorrect estimates of dividend growth, this process generates the “bounded rational risk” which is captured by  $\Phi$ .

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<sup>1</sup> Theoretically, Brennan (1998) shows that “learning risk” affects investors’ investments in stock. The literature of learning also includes Detemple (1986), Veronesi (1999, 2000), Brennan and Xia (2001), Buraschi and Jiltsov (2006), David (2008), and Xiong and Yan (2009). Empirically, Pastor and Veronesi (2003) investigate the impact of learning on stock profitability to explain the market-to-book premium. Massa and Simonov (2005) assess how the uncertainty induced by investors’ learning about fundamental variables affects stock returns, and provide a literature review on “learning risk.” Lundtofe (2006) analyzes the effect of differences in information quality among investors on portfolio choice, in which investors have different information sets.

## 2.4 Security market

Investors can trade continuously in three financial assets: a riskless bond  $B$ , a stock  $S$ , and a long term bond  $B_T$ . The riskless bond is in zero net supply and pays the real interest rate  $r$ , the stock is in net supply of 1 and represents a claim on the aggregate dividends  $D$ , and the long term bond has payoff 1 at maturity  $T$  with price dynamics

$$dB(t) = r(t)B(t)dt, \quad (7)$$

$$\frac{dS(t) + D(t)dt}{S(t)} = \mu_S(t)dt + \sigma_S(t)^T dW(t) = \mu_S^i(t)dt + \sigma_S(t)^T dW^i(t), \quad (8)$$

$$\frac{dB_T(t)}{B_T(t)} = \mu_B(t)dt + \sigma_B(t)^T dW(t) = \mu_B^i(t)dt + \sigma_B(t)^T dW^i(t). \quad (9)$$

Equations (8) and (9) represent the asset dynamics perceived by investors  $i$ . From market completeness, there exists a unique state price density  $\zeta^i$  for each type of investors. Under the no arbitrage assumption, it follows that

$$d\zeta^i(t) = -\zeta^i(t) \left[ r(t)dt + \theta_1^i(t)dW_1^i(t) + \theta_2^i(t)dW_2^i(t) \right], \quad i = A, B, \quad (10)$$

where  $\theta^i(t) = (\theta_1^i(t), \theta_2^i(t))$  is the market price of risk perceived by investor  $i$ .

## 2.5 Investors' preferences and optimization

Investor  $i$  is endowed, at time  $t$ , with  $\alpha^i(t)$  shares of stock where  $\alpha^i(t) > 0$  and  $\alpha^A(t) + \alpha^B(t) = 1$ . He chooses a nonnegative consumption process  $c^i$  and a portfolio process  $\pi^i = (\pi_S^i, \pi_B^i)$ . The dynamic budget constraint satisfies ( $i = A, B$ )

$$dX^i(t) = \left[ X^i(t)r(t) - c^i(t) \right] dt + X^i(t)\pi_S^i(t) \left[ (\mu_S^i(t) - r(t))dt + \sigma_S(t)^T dW^i(t) \right] \\ + X^i(t)\pi_B^i(t) \left[ (\mu_B^i(t) - r(t))dt + \sigma_B(t)^T dW^i(t) \right] \quad (11)$$

Following the martingale techniques (e.g., Karatzas, Lehoczky and Shreve, 1987; Cox and Huang, 1989), investor  $i$ 's dynamic consumption portfolio problem can be converted into the following static problem:

$$\max_{c^i} E_t^i \left[ \int_t^T \frac{c^i(s)^{1-\gamma}}{1-\gamma} ds \right] \quad s.t. \quad E_t^i \left[ \int_t^T \zeta^i(s) c^i(s) ds \right] \leq \zeta^i(t) \alpha^i(t) S(t), \quad (12)$$

where  $E_t^i[\cdot]$  denotes the expectation conditional on the information structure  $(\Omega, F^i, P^i)$ .

Equation (12) takes into account, in the left hand side, the present value of investor  $i$ 's lifetime consumption, and his endowed wealth appears in the right side.

## 2.6 Equilibrium

**Definition:** An equilibrium is a price system  $(r, S, B_T)$  and an admissible consumption portfolio process  $(c^i, \pi^i)$  such that (i) investors choose their optimal consumption and portfolio strategies given their perceived price processes; (ii) asset prices are consistent across investors, that is,

$$\begin{pmatrix} \mu_S^B(t) - \mu_S^A(t) \\ \mu_B^B(t) - \mu_B^A(t) \end{pmatrix} = \begin{pmatrix} \sigma_S(t) \\ \sigma_B(t) \end{pmatrix} \Phi(t), \quad (13)$$

and (iii) markets of goods and assets clear, that is,

$$c^A(t) + c^B(t) = D(t), \quad (14)$$

$$\begin{aligned} \pi_S^A(t) X^A(t) + \pi_S^B(t) X^B(t) &= S(t), \\ X^A(t) + X^B(t) &= S(t), \\ \pi_B^A(t) X^A(t) + \pi_B^B(t) X^B(t) &= 0. \end{aligned} \quad (15)$$

The representative agent is endowed with the aggregate dividends and with utility function (Cuoco and He, 1994; Basak and Cuoco, 1998)

$$U(c(t); \lambda(t)) \equiv \max_{c^A(t)+c^B(t)=c(t)} \frac{c^A(t)^{1-\gamma}}{1-\gamma} + \lambda(t) \frac{c^B(t)^{1-\gamma}}{1-\gamma}, \quad (16)$$

where  $\lambda > 0$  denotes the relative weight of investors  $B$ , and can be stochastic over time.

### 3. Benchmark economy without bounded rational investors

If investor  $B$  is also rational in processing the signal but has different prior belief about dividend growth from investor  $A$ , the economy becomes a standard one with heterogeneous beliefs (e.g., Basak, 2000; Buraschi and Jiltsov, 2006).

**Proposition 1:** *In Equilibrium, the relative weight  $\lambda$  satisfies*

$$d\lambda(t) = -\lambda(t) \Phi^{(B)}(t)^T dW^A(t), \quad (17)$$

where  $\Phi^{(B)} \equiv (\Phi_1^{(B)}, \Phi_2^{(B)})^T$  satisfies  $\Phi_1^{(B)}(t) \equiv \kappa_2 (\mu_D^A(t) - \mu_D^B(t))$  and  $\Phi_2^{(B)}(t) \equiv -\frac{\sigma_{e1}}{\sigma_{e2}} \Phi_1^{(B)}$ .

The optimal consumptions are

$$c^A(t) = (1 - \omega(t))D(t), \quad c^B(t) = \omega(t)D(t), \quad (18)$$

where  $\omega(t) = \frac{\lambda(t)^{1/\gamma}}{1 + \lambda(t)^{1/\gamma}}$  is investor  $B$ 's consumption share.

The stock price and the long term bond price are equal to

$$S(t) = E_t^i \left[ \int_t^T \frac{\zeta^i(s)}{\zeta^i(t)} D(s) ds \right], \quad B_T(t) = E_t^i \left[ \frac{\zeta^i(T)}{\zeta^i(t)} \right], \quad (19)$$

which is the standard expression for asset prices because the stock is a claim on the aggregate dividends  $D$  and the long term bond has payoff 1 at maturity time  $T$ . I use the Clark-Ocone formula and the martingale representation theorem (e.g., Ocone and Karatzas, 1991) to decompose the diffusion items of assets.

**Proposition 3:** *The diffusion coefficients of the stock price and the long term bond price are as follows*

$$\begin{aligned}\sigma_{s1}(t) &= \sigma_{D1} + w_s(t)\Phi_1^{(B)}(t) + \frac{(1-\gamma)\kappa_2\nu(t)}{S(t)} E_t^A \left[ \int_t^T (s-t) \frac{\zeta^A(s)}{\zeta^A(t)} D(s) ds \right], \\ \sigma_{s2}(t) &= \sigma_{D2} + w_s(t)\Phi_2^{(B)}(t),\end{aligned}\quad (20)$$

and

$$\begin{aligned}\sigma_{B1}(t) &= w_B(t)\Phi_1^{(B)}(t) - \frac{\gamma\kappa_2\nu(t)}{B_T(t)} E_t^A \left[ (T-t) \frac{\zeta^A(T)}{\zeta^A(t)} \right], \\ \sigma_{B2}(t) &= w_B(t)\Phi_2^{(B)}(t),\end{aligned}\quad (21)$$

where  $w_s(t) = \frac{\lambda(t)^{1/\gamma}}{1 + \lambda(t)^{1/\gamma}} - \frac{X^B(t)}{S(t)}$ ,  $w_B(t) = \frac{\lambda(t)^{1/\gamma}}{1 + \lambda(t)^{1/\gamma}} - \frac{1}{B_T(t)} E_t^A \left[ \frac{\lambda(T)^{1/\gamma}}{1 + \lambda(T)^{1/\gamma}} \frac{\zeta^A(T)}{\zeta^A(t)} \right]$ ,

and  $X^B(t) = E_t^B \left[ \int_t^T \frac{\zeta^B(s)}{\zeta^B(t)} c^B(s) ds \right]$  is investor B's wealth at time  $t$ .

The diffusion item of stock,  $\sigma_{s1}$ , is composed of three components. The first term is the dividend volatility, the second term is the effect of the shock  $dW_1$  on current and future cash flows, and the third term captures the effect of “learning risk” caused by rational investors' estimates of unobserved dividend growth.

#### 4. Economy with bounded rational investors

**Proposition 4:** *In Equilibrium, the relative weight  $\lambda$  satisfies*

$$d\lambda(t) = -\lambda(t)\Phi^{(M)}(t)^T dW^A(t), \quad (22)$$

where  $\Phi^{(M)} \equiv (\Phi_1^{(M)}, \Phi_2^{(M)})^T$  satisfies  $\Phi_1^{(M)}(t) = \kappa_2(\mu_D^A(t) - \mu_e)$  and  $\Phi_2^{(M)}(t) = -\frac{\sigma_{e1}}{\sigma_{e2}}\Phi_1^{(M)}$ .

The optimal consumptions are

$$c^A(t) = (1 - \omega(t))D(t), \quad c^B(t) = \omega(t)D(t), \quad (23)$$

where  $\omega(t) = \frac{\lambda(t)^{1/\gamma}}{(1 + \lambda(t)^{1/\gamma})}$  is bounded rational investor B's consumption share.

Equation (22) shows that  $\lambda$  plays an important role in determining optimal consumption: the higher  $\lambda$  is, the higher the wealth allocated to the bounded rational investor and thus the higher his optimal consumption.

Using the Clark-Ocone formula and the martingale representation theorem (e.g., Ocone and Karatzas, 1991), I can calculate the diffusion items of assets.

**Proposition 6:** *The diffusion coefficients of the stock price and the long-term bond price are given follows*

$$\begin{aligned} \sigma_{s_1}(t) &= \sigma_{D_1} + w_S(t)\Phi_1^{(M)}(t) + \frac{(1-\gamma)\kappa_2\nu(t)}{S(t)} E_t^A \left[ \int_t^T (s-t) \frac{\zeta^A(s)}{\zeta^A(t)} D(s) ds \right] \\ &\quad - \frac{1}{S(t)} E_t^A \left[ \int_t^T f(t,s) \frac{\zeta^A(s)}{\zeta^A(t)} c^B(s) ds \right], \\ \sigma_{s_2}(t) &= \sigma_{D_2} + w_S(t)\Phi_2^{(M)}(t), \end{aligned} \quad (24)$$

and

$$\begin{aligned} \sigma_{B_1}(t) &= w_B(t)\Phi_1^{(M)}(t) - \frac{\gamma\kappa_2\nu(t)}{B_T(t)} E_t^A \left[ (T-t) \frac{\zeta^A(T)}{\zeta^A(t)} \right] - \frac{1}{B_T(t)} E_t^A \left[ f(t,T) w(T) \frac{\zeta^A(T)}{\zeta^A(t)} \right], \\ \sigma_{B_2}(t) &= w_B(t)\Phi_2^{(M)}(t), \end{aligned} \quad (25)$$

$$\text{where } f(t,s) = \kappa_2^2 \nu(t) \left[ \int_t^s \left[ 1 + \left( \frac{\sigma_{e1}}{\sigma_{e2}} \right)^2 \right] \Phi_1^{(M)}(u) du + \int_t^s dW_1^A(s) - \int_t^s \frac{\sigma_{e1}}{\sigma_{e2}} dW_2^A(s) \right].$$

The diffusion item of stock price  $\sigma_{s_1}$  consists of four terms. The first term is the dividend volatility, and the second term is the effect of the shock  $dW_1$  on current and future cash. The two terms are standard ones and consistent with the literature. The third

term captures the effect of “learning risk” caused by rational investor, and the last term reflects the effect of “bounded rational risk” generated by the bounded rational investor.

Veronesi (1999) and David and Veronesi (2008) find that investors’ uncertainty about fundamentals tends to increase stock volatility. It means that the “*learning risk*” causes the stock volatility to increase. The “*bounded rational risk*” is generated by investors’ bounded rational behavior in processing information. Although I cannot provide a rigorous proof, the simulation results suggest that “bounded rational risk” tends to increase the stock volatility.

Table 1

**Parameter values for numerical analysis**

	Value
Growth rate of dividends	$\mu_D=3\%$
Volatility of dividend growth caused by $dW_1$	$\sigma_{D1}=3.5\%$
Volatility of dividend growth caused by $dW_2$	$\sigma_{D2}=3.5\%$
Drift of signal process	$2\% \leq \mu_e \leq 4\%$
Volatility of signal process caused by $dW_1$	$\sigma_{e1}=0$
Volatility of signal process caused by $dW_2$	$\sigma_{e2}=3.5\%$
Coefficient of relative risk aversion	$\gamma=3$
Investors’ investment horizon	$T=30$ years
Frequency of investors’ belief updating (year)	$\Delta t=1/52$ year
Investor A's initial estimate of dividend growth	$\mu_D^A(0)=3\%$
Initial variance of investor A's estimates of dividend growth	$v(0)=0.09\%$

## 5. Numerical analysis

Using the Monte Carlo simulation, I numerically investigate the implications of investors' bounded rational behavior on stock price, stock volatility, stock-bond return correlation, and rational investor’s welfare changes. Table 1 illustrates all parameters (e.g., Buraschi and Jiltsov (2006)) for simulation analysis. I simulate one million paths of

equilibrium variables using the Milstein discretization scheme and compare the model with the benchmark.<sup>2</sup>

### 5.1 Stock price

Figure 1 plots the stock price against the difference between the drift of signal process and dividend growth,  $\mu_e - \mu_D$ . I find that the stock price is higher than that in the benchmark when the drift of signal process exceeds dividend growth. It is intuitive: when the drift of signal process is higher than dividend growth, the bounded rational investor is more optimistic than the rational investor because his or her estimate of dividend growth equals the drift of signal process, and in turn makes him or her perceive the stock more favorably and thus causes the stock price to increase.

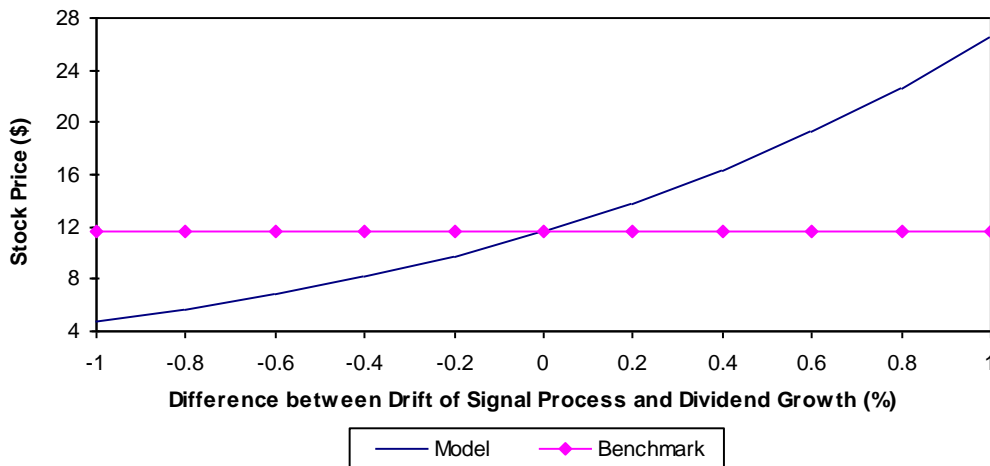


Figure 1

#### **The impact of investors' bounded rational behavior on stock price**

The graph plots the stock price against the difference between the drift of signal process and dividend growth. .

<sup>2</sup> It doesn't change the results qualitatively for bond market, and the results can be obtained from the author on request.

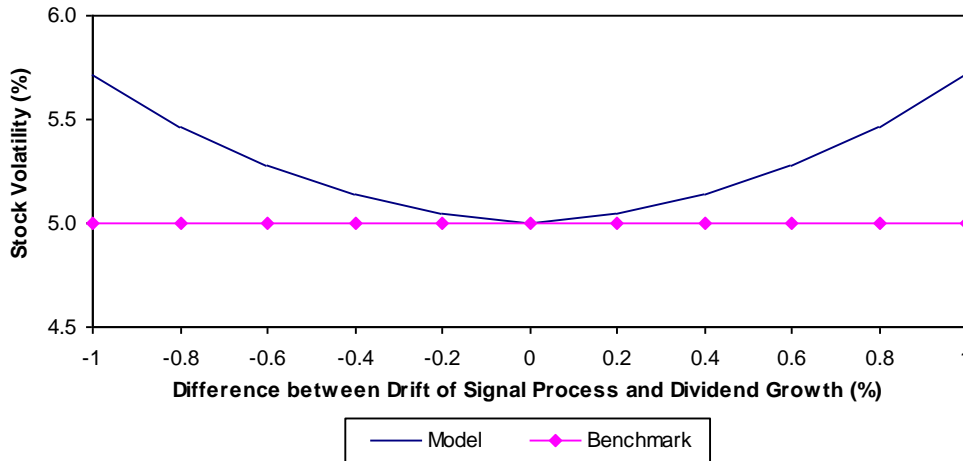


Figure 2

**The impact of investors' bounded rational behavior on stock volatility**

The graph plots the stock volatility against the difference between the drift of signal process and dividend growth.

*5.2 Stock volatility*

Figure 2 illustrates the stock volatility against the difference between the drift of signal process and dividend growth, and demonstrates that stock volatility in the model is higher than that in the benchmark. Both “learning risk” and “bounded rational risk” affect stock volatility, making stock prices more volatile than they would be in the benchmark. The result can help explain the “excessive volatility puzzle.”

*5.3 Correlation between stock and bond returns*

Figure 3 shows the stock bond return correlation against the difference between the drift of signal process and dividend growth. I find that the stock bond return correlation decreases significantly compared with the benchmark, and becomes negative as the difference between the drift of signal process and dividend growth becomes greater than

0.6%. The result implies that, as the new information arrives, the drift of signal process changes over time, and thus the correlation becomes time varying and even negative.

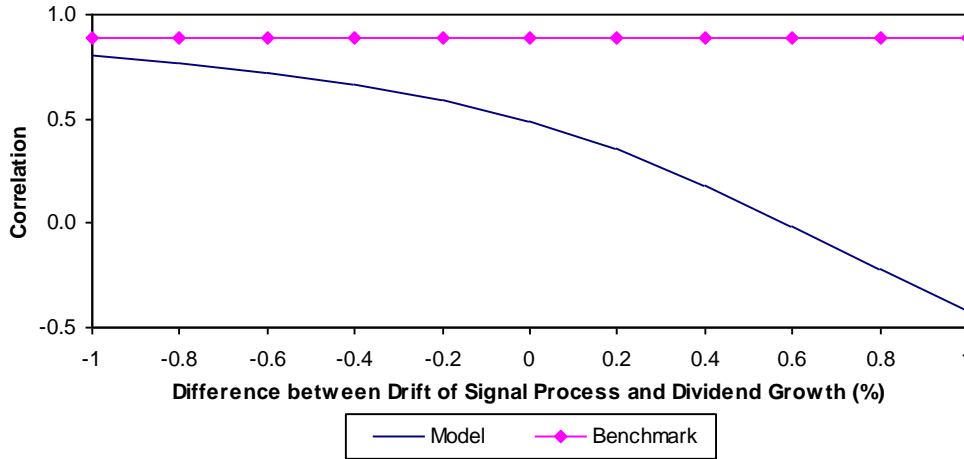


Figure 3

**The impact of investors' bounded rational behavior on stock-bond return correlation**

The graph plots the stock bond return correlation against the difference between the drift of signal process and dividend growth.

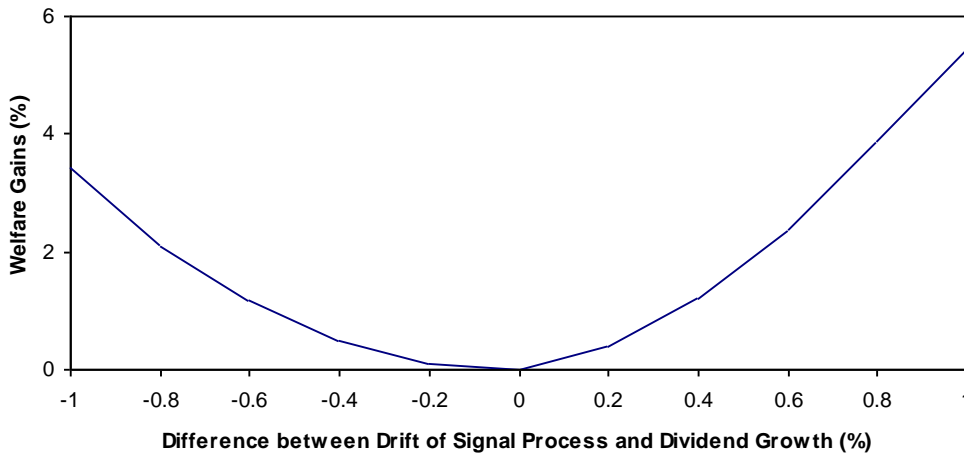


Figure 4

**The impact of investors' bounded rational behavior on rational investors' welfare**

The graph plots rational investors' welfare gains against the difference between the drift of signal process and dividend growth.

*5.4 Welfare analysis*

In the presence of bounded rational investors, rational investors' optimal consumption changes, and so does their welfare. In this section, I explore rational

investors' welfare change, which is defined as the percentage change in their utility compared with the benchmark.

Figure 4 plots rational investors' welfare against the difference between the drift of signal process and dividend growth. It demonstrates that rational investors gain from the trading with bounded rational investors. It is intuitive: rational investors are correct in processing information and thus have more accurate estimates about dividend growth than bounded rational investors. Therefore, they can benefit from the trading with bounded rational investors.

## **6. Conclusion**

In an incomplete information setting, I investigate the impact of investors' bounded rational behavior on asset pricing. The bounded rational investors mistakenly believe that the dividend growth equals to the drift of signal process. This process generates "bounded rational risk." I find that the volatility of stock and bond returns is increased in the presence of bounded rational investors, the stock-bond return correlation becomes time varying and even negative, and rational investors benefit from trading with bounded rational investors.

## Appendix

**Proof for Propositions 1 and 4:** The proof is an adaptation of Karatzas, Lehoczky and Shreve (1987) and Cox and Huang (1989).

**Proof for Propositions 2 and 5:** The expressions follow from applying Ito's lemma to the state price density.

**Proof for Proposition 3:** The method is the same as the proof for Proposition 6 by replacing  $\Phi^{(M)}$  with  $\Phi^{(B)}$ .

**Proof for Proposition 6:** Let  $M(t)$  be the  $\mathcal{L}^2(\mathbb{P}_1)$ -martingale

$$M(t) = E_t^A \left[ \int_0^T D(s)^{1-\gamma} \left(1 + \lambda(s)^{1/\gamma}\right)^\gamma ds \right]. \quad (\text{A.1})$$

Following the martingale representation theorem,

$$M(t) = M(0) + \int_0^t \left[ \phi_1(s) dW_1^A(s) + \phi_2(s) dW_2^A(s) \right], \quad (\text{A.2})$$

where  $\phi_j$  ( $j=1,2$ ) is the optional projection of the derivative of  $M(t)$ . Therefore, stock price is given by

$$S(t) = \left( \frac{D(t)}{1 + \lambda(t)^{1/\gamma}} \right)^\gamma \left[ M(t) - \int_0^t \left( \frac{D(s)}{1 + \lambda(s)^{1/\gamma}} \right)^{-\gamma} D(s) ds \right]. \quad (\text{A.3})$$

Applying Ito's lemma to equation (A.3) and matching the diffusion coefficients with the dynamics of  $S$ , the diffusion coefficients of stock price are given by ( $j=1,2$ ),

$$\sigma_{S_j}(t) = \gamma \sigma_{D_j} + \frac{\lambda(t)^{1/\gamma}}{1 + \lambda(t)^{1/\gamma}} \Phi_j^{(M)} + \frac{\phi_j(t)}{\zeta^A(t) S(t)}. \quad (\text{A.4})$$

Following the Clark-Ocone formula and the Malliavin derivatives (Nualart, 1995), I have

$$\phi_j(t) = E_t^A \left[ \int_t^T D(s)^{1-\gamma} \left(1 + \lambda(s)^{1/\gamma}\right)^\gamma \left[ \frac{\lambda(s)^{1/\gamma}}{1 + \lambda(s)^{1/\gamma}} \frac{D_t^j \lambda(s)}{\lambda(s)} + (1-\gamma) \frac{D_t^j D(s)}{D(s)} \right] ds \right]. \quad (\text{A.5})$$

Similarly, the diffusion coefficients of bond price are given by

$$\sigma_{B_j}(t) = \gamma \sigma_{D_j} + \frac{\lambda(t)^{1/\gamma}}{1 + \lambda(t)^{1/\gamma}} \Phi_j^{(M)} + \frac{\varphi_j(t)}{\zeta^A(t) B_T(t)}, \quad (\text{A.6})$$

$$\varphi_j(t) = E_t^A \left[ D(T)^{1-\gamma} \left(1 + \lambda(T)^{1/\gamma}\right)^\gamma \left[ \frac{\lambda(T)^{1/\gamma}}{1 + \lambda(T)^{1/\gamma}} \frac{D_t^j \lambda(T)}{\lambda(T)} - \gamma \frac{D_t^j D(T)}{D(T)} \right] \right]. \quad (\text{A.7})$$

I solve for equations (1) and (22) to derive the following Malliavin derivatives  $D_t^1 D(s) = [\sigma_{D1} + \kappa_2 v(t)(s-t)]D(s)$ ,  $D_t^2 D(s) = \sigma_{D2} D(s)$ ,  $D_t^1 \lambda(s) = -[f(t,s) + \Phi_1^{(M)}(t)]\lambda(s)$ , and  $D_t^2 \lambda(s) = -\Phi_2^{(M)}(t)\lambda(s)$ . Rearranging equations (A.4)-(A.7), I can derive the diffusion coefficients of stock and bond prices.

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